Reduction formula

1. **(a)** **(i)** $I_0 = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}\Big|_0^1 = \frac{\pi}{2}$

$$J_0 = \int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}} \underset{x=\sin\theta}{=} \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin^2\theta} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{2\sin^2\theta+\cos^2\theta} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{2\tan^2\theta+1} \underset{u=\tan\theta}{=} \int_0^{\infty} \frac{du}{2u^2+1}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\infty} \frac{d(\sqrt{2}u)}{(\sqrt{2}u)^2+1} = \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}u)\Big|_0^{\infty} = \frac{\pi}{2\sqrt{2}} \qquad \therefore \quad J_0 = \frac{\sqrt{2}}{2} I_0$$

(ii) Let $u = x^{2n-1}$, $du = (2n-1)x^{2(n-1)}dx$, $dv = \frac{xdx}{\sqrt{1-x^2}}$, $v = -\sqrt{1-x^2}$

$$I_n = \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}dx = x^{2n-1}\left(-\sqrt{1-x^2}\right)\Big|_0^1 - \int_0^1 \left(-\sqrt{1-x^2}\right)\left[(2n-1)x^{2(n-1)}dx\right]$$

$$= (2n-1)\int_0^1 x^{2(n-1)}\sqrt{1-x^2}\,dx = (2n-1)\int_0^1 x^{2(n-1)}\frac{(1-x^2)}{\sqrt{1-x^2}}dx = (2n-1)[I_{n-1} - I_n]$$

$\therefore \quad 2nI_n = (2n-1)\, I_{n-1}$.

(b) $J_n + J_{n-1} = I_{n-1} \qquad \therefore \quad J_n = I_{n-1} - J_{n-1}$

(c) $J_3 = I_2 - J_2 = \frac{3}{4}I_1 - (I_1 - J_1) = J_1 - \frac{1}{4}I_1 = (I_0 - J_0) - \frac{1}{4}\left(\frac{1}{2}I_0\right) = \frac{7}{8}I_0 - J_0 = \frac{7}{8}\left(\frac{\pi}{2}\right) - \frac{\sqrt{2}}{4}\pi = \frac{\pi(7-4\sqrt{2})}{16}$

2. $u_n = \int_0^{\pi/2} x\cos^n x\,dx = \int_0^{\pi/2} x\cos^{n-1}x\,d\sin x = x\cos^{n-1}x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x\left[\cos^{n-1}x - (n-1)x\cos^{n-2}x\sin x\right]dx$

$$= 0 - \int_0^{\pi/2} \cos^{n-1}x\sin x\,dx + (n-1)\int_0^{\pi/2} \cos^{n-2}x\sin^2 x\,dx = -\int_0^{\pi/2}\cos^{n-1}x\,d(\cos x) + (n-1)\int_0^{\pi/2}\cos^{n-2}x(1-\cos^2 x)dx$$

$$= \frac{\cos^n x}{n}\Big|_0^{\pi/2} + (n-1)(u_{n-2} - u_n) = -\frac{1}{n} + (n-1)(u_{n-2} - u_n)$$

$\therefore \quad u_n = -\frac{1}{n^2} + \frac{n-1}{n}u_{n-2}$

$$u_4 = -\frac{1}{16} + \frac{3}{4}u_2 = -\frac{1}{16} + \frac{3}{4}\left(-\frac{1}{4} + \frac{1}{2}u_0\right) = -\frac{1}{4} + \frac{3}{8}u_0 = -\frac{1}{4} + \frac{3}{8}\int_0^{\pi/2} x\,dx = -\frac{1}{4} + \frac{3}{8}\frac{\pi^2}{8} = \frac{3\pi^2 - 16}{64}$$

$$u_5 = -\frac{1}{25} + \frac{4}{5}u_3 = -\frac{1}{25} + \frac{4}{5}\left(-\frac{1}{9} + \frac{2}{3}u_1\right) = -\frac{29}{225} + \frac{8}{15}u_1 = -\frac{29}{225} + \frac{8}{15}\int_0^{\pi/2} x\cos x\,dx$$

$$= -\frac{29}{225} + \frac{8}{15}\int_0^{\pi/2} x\,d(\sin x) = -\frac{29}{225} + \frac{8}{15}\left[x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx\right] = -\frac{29}{225} + \frac{8}{15}\left[\frac{\pi}{2} - 1\right] = \frac{60\pi - 149}{225}$$

3. $I_n = \int \frac{x^n}{\sqrt{a^2+x^2}}dx = \int x^{n-1}\frac{x}{\sqrt{a^2+x^2}}dx = \int x^{n-1}d\sqrt{a^2+x^2} = x^{n-1}\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\,(n-1)x^{n-2}dx$

$$= x^{n-1}\sqrt{a^2+x^2} - (n-1)\int \frac{a^2+x^2}{\sqrt{a^2+x^2}}x^{n-2}dx = x^{n-1}\sqrt{a^2+x^2} - (n-1)\left[a^2 I_{n-2} + I_n\right]$$

Solve for I_n, $\therefore \quad I_n = \frac{x^{n-1}}{n}\sqrt{a^2+x^2} - \frac{n-1}{n}a^2 I_{n-2}$

Put $a^2 = 5$, $I_5 = \int_0^2 \frac{x^5}{\sqrt{5+x^2}}dx = \frac{48}{5} - 4I_3 = \frac{48}{5} - 4\left(4 - \frac{10}{3}I_1\right)$

$I_1 = \int_0^2 \frac{x}{\sqrt{5+x^2}}dx = \int_0^2 \frac{d\sqrt{5+x^2}}{\sqrt{5+x^2}} = \sqrt{5+x^2}\Big|_0^2 = 3-\sqrt{5}$, $\therefore \quad I_5 = \underline{\underline{\frac{168}{5} - \frac{40}{3}\sqrt{5}}}$

4. $u_m = \int x^m(a^2-x^2)^{1/2}dx = \int x^{m-1}\left[x(a^2-x^2)^{1/2}dx\right] = -\frac{1}{3}\int x^{m-1}(a^2-x^2)^{3/2}dx$

$3u_m = -x^{m-1}(a^2-x^2)^{3/2} + \int (a^2-x^2)^{3/2}(m-1)x^{m-2}dx = -x^{m-1}(a^2-x^2)^{3/2} + (m-1)\int (a^2-x^2)^{1/2}\left[(a^2-x^2)x^{m-2}\right]dx$

$= -x^{m-1}(a^2-x^2)^{3/2} + (m-1)\left[a^2u_{m-2} - u_m\right]$

$\therefore \quad (m+2)\,u_m = -x^{m-1}(a^2-x^2)^{3/2} + a^2(m-1)\,u_{m-2}$.

Put $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $a = 1$ and replace m by $2m$.

$u_{2m} = \int_0^{\pi/2} \sin^{2m}\theta\cos^2\theta d\theta = \int_0^1 x^{2m}(1-x^2)^{1/2}dx = \frac{1}{2m+2}\left[-x^{m-1}(1-x^2)^{3/2}\Big|_0^1 + (2n-1)u_{2m-2}\right] = \frac{2m-1}{2m+2}u_{2m-2}$

$= \frac{2m-1}{2m+2}\frac{2m-3}{2m}u_{2m-4} = \ldots = \frac{(2m-1)(2m-3)\ldots 3\cdot 1}{(2m+2)(2m)\ldots 6\cdot 4}u_0 = \frac{(2m-1)(2m-3)\ldots 3\cdot 1}{(2m+2)(2m)\ldots 6\cdot 4\cdot 2}\times\frac{\pi}{2}$

since $u_0 = \int_0^{\pi/2} \sin^0\theta\cos^2\theta d\theta = \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)d\theta = \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{4} = \frac{1}{2}\frac{\pi}{2}$.

5. $\frac{d}{dx}x^{n-1}(2ax-x^2)^{\frac{3}{2}} = x^{n-1}\frac{3}{2}(2ax-x^2)^{\frac{1}{2}}(2a-2x) + (n-1)x^{n-2}(2ax-x^2)^{\frac{3}{2}}$

$= 3x^{n-1}(a-x)(2ax-x^2)^{\frac{1}{2}} + (n-1)x^{n-2}(2ax-x^2)(2ax-x^2)^{\frac{1}{2}}$

$= 3ax^{n-1}(2ax-x^2)^{\frac{1}{2}} - 3x^n(2ax-x^2)^{\frac{1}{2}} + 2a(n-1)x^{n-1}(2ax-x^2)^{\frac{1}{2}} - (n-1)x^n(2ax-x^2)^{\frac{1}{2}}$

$= (2n+1)ax^{n-1}(2ax-x^2)^{\frac{1}{2}} - (n+2)x^n(2ax-x^2)^{\frac{1}{2}}$

Integrate both sides, $x^{n-1}(2ax-x^2)^{\frac{3}{2}} = (2n+1)au_{n-1} - (n+2)u_n$. Result follows.

$u_0 = \int_0^{2a}(2ax-x^2)^{1/2}dx = \int_0^{2a}\sqrt{a^2-(x-a)^2}dx \underset{x-a=a\sin\theta}{=} a^2\int_{-\pi/2}^{\pi/2}\cos^2\theta d\theta = \underline{\underline{\frac{1}{2}\pi a^2}}$

$u_2 = \int_0^{2a} x^2(2ax-x^2)^{1/2}dx = \frac{5}{4}au_1 - \left[\frac{1}{4}x(2ax-x^2)^{\frac{3}{2}}\right]_0^{2a} = \frac{5}{4}au_1 = \frac{5}{4}a\left[\frac{3}{3}au_0\right] = \frac{5}{4}a^2u_0 = \underline{\underline{\frac{5}{8}\pi a^4}}$

6. $I_n = \int_0^a \frac{x^n}{\sqrt{3a^2+x^2}}dx = \int_0^a x^{n-1}d\left(\sqrt{3a^2+x^2}\right) = x^{n-1}\sqrt{3a^2+x^2}\Big|_0^a - \int_0^a \sqrt{3a^2+x^2}(n-1)x^{n-2}dx$

$= 2a^n - (n-1)\int_0^a \frac{3a^2+x^2}{\sqrt{3a^2+x^2}}dx = 2a^n - (n-1)\left[3a^2I_{n-2} + I_n\right]$ $\therefore \quad I_n = \frac{2a^n}{n} - \frac{3(n-1)}{n}a^2I_{n-2}$

$I_7 = \frac{2a^7}{7} - \frac{18}{7}a^2I_5 = -\frac{26}{35}a^7 + \frac{216}{35}a^4I_3 = \frac{118}{35}a^7 - \frac{432}{35}a^6I_1 = \frac{118}{35}a^7 - \frac{432}{35}a^6\left[2a-\sqrt{3}a\right] = \underline{\underline{\frac{4}{35}a^7\left(108\sqrt{3}-137\right)}}$

7. $I_{p,q} = \int \frac{x^p}{(1+x^2)^q}dx = -\frac{1}{2(q-1)}\int x^{p-1}d\left[\frac{1}{(1+x^2)^{q-1}}\right]$

$$2(q-1)I_{p,q} = -\frac{x^{p-1}}{(1+x^2)^{q-1}} + \int \frac{1}{(1+x^2)^{q-1}}(p-1)x^{p-2}dx = -\frac{x^{p-1}}{(1+x^2)^{q-1}} + (p-1)I_{p-2,q-1}$$

$$\int_0^1 \frac{x^6}{(1+x^2)^3}dx = I_{6,3} = -\frac{1}{4}\frac{x^5}{(1+x^2)^2}\Big|_0^1 + \frac{5}{4}I_{4,2} = -\frac{1}{16}+\frac{5}{4}I_{4,2} = -\frac{3}{8}+\frac{15}{8}I_{2,1} = \underline{\underline{-\frac{3}{2}-\frac{15}{32}\pi}}$$

where $I_{2,1} = \int_0^1 \frac{x^2}{(1+x^2)^3}dx = \int_0^1\left[1-\frac{1}{1+x^2}\right]dx = \left[x-\tan^{-1}x\right]_0^1 = 1-\frac{\pi}{4}$

8. $I_{n+1} = \int_0^{2a} x^{n+1}\sqrt{2ax-x^2}dx = \int_0^{2a} x^n[(x-a)+a]\sqrt{2ax-x^2}dx = \int_0^{2a} x^n(x-a)\sqrt{2ax-x^2}dx + a\int_0^{2a} x^n\sqrt{2ax-x^2}dx$

$$= \int_0^{2a} x^n d\left[-\frac{1}{3}(2ax-x^2)^{3/2}\right] + aI_n = -\frac{1}{3}(2ax-x^2)^{3/2}x^n\Big|_0^{2a} + \frac{1}{3}\int_0^{2a} nx^{n-1}(2ax-x^2)^{3/2}dx + aI_n$$

$$= 0 + \frac{n}{3}\int_0^{2a} x^{n-1}(2ax-x^2)(2ax-x^2)^{1/2}dx + aI_n = \frac{n}{3}[2aI_n - I_{n+1}] + aI_n \qquad \therefore \quad 3aI_n - 3I_{n+1} = n(I_{n+1} - 2a I_n).$$

$$\therefore \quad I_{n+1} = a\frac{2n+3}{n+3}I_n \Rightarrow I_n = a\frac{2n+1}{n+2}I_{n-1} = a^2\frac{2n+1}{n+2}\frac{2n-1}{n+1}I_{n-2} = \ldots = a^n\frac{(2n+1)(2n-1)\ldots7\cdot5\cdot3}{(n+2)(n+1)\ldots5\cdot4\cdot3}I_0$$

$$= \underline{\underline{\frac{(2n+1)(2n-1)\ldots7\cdot5}{(n+2)(n+1)\ldots5\cdot4}\times\frac{\pi}{2}a^{n+2}}}$$

since $I_0 = \int_0^{2a}(2ax-x^2)^{1/2}dx = \int_0^{2a}\sqrt{a^2-(x-a)^2}dx \underset{x-a=a\sin\theta}{=} a^2\int_{-\pi/2}^{\pi/2}\cos^2\theta d\theta = \underline{\underline{\frac{1}{2}\pi a^2}}$

9. $I_n(z) = \int_0^1 (1-y)^n(e^{yz}-1)dy = \int_0^1 (1-y)^n e^{yz}dy - \int_0^1 (1-y)^n dy = \frac{1}{z}\int_{y=0}^{y=1}(1-y)^n d(e^{yz}) - \int_{y=0}^{y=1}(1-y)^n d(1-y)$

$$= \frac{1}{z}\left[(1-y)^n e^{yz}\Big|_0^1 - \int_0^1 (-n)(1-y)^{n-1}e^{yz}dy\right] - \frac{(1-y)^{n+1}}{n+1}\Big|_0^1 = \frac{1}{z}\left[-1+n\int_0^1(1-y)^{n-1}(e^{yz}-1+1)dy\right] - \frac{1}{n+1}$$

$$= \frac{1}{z}\left[-1+nI_{n-1}(z) - (1-y)^n\Big|_0^1\right] - \frac{1}{n+1} = \frac{1}{z}nI_{n-1}(z) - \frac{1}{n+1} \qquad \therefore \quad I_{n-1}(z) = \frac{z}{n}I_n(z) + \frac{z}{n(n+1)}, \quad n \ge 1$$

$I_0 = \frac{z}{1}I_1 + \frac{z}{1\times2}$	(1)	(1)×1,	$I_0 = \frac{z}{1!}I_1 + \frac{z}{2!}$
$I_1 = \frac{z}{2}I_2 + \frac{z}{2\times3}$	(2)	$(2)\times\frac{z}{1!}$,	$\frac{z}{1!}I_1 = \frac{z^2}{2!}I_2 + \frac{z^2}{3!}$
$I_2 = \frac{z}{3}I_3 + \frac{z}{3\times4}$	(3)	$(3)\times\frac{z^2}{2!}$,	$\frac{z^2}{2!}I_2 = \frac{z^3}{3!}I_3 + \frac{z^3}{4!}$
…..	:	….	
$I_{n-2} = \frac{z}{n-1}I_3 + \frac{z}{(n-1)\times n}$	(n)	$(n)\times\frac{z^{n-2}}{(n-2)!}$,	$\frac{z^{n-2}}{(n-2)!}I_{n-2} = \frac{z^{n-1}}{(n-1)!}I_3 + \frac{z^{n-1}}{n!}$

Adding the last sets of equalities and cancelling terms, $I_0 = \sum_{r=2}^{n} \frac{z^{r-1}}{r!} + \frac{z^{n-1}}{(n-1)!} I_{n-1}(z)$ ….(1)

But $I_0(z) = \int_0^1 (e^{yz} - 1)dy = \left[\frac{e^{yz}}{z} - y\right]_0^1 = \frac{e^z}{z} - \frac{1}{z} - 1$ ….(2)

Equating (1) and (2), $\frac{e^z - 1 - z}{z} = \sum_{r=2}^{n} \frac{z^{r-1}}{r!} + \frac{z^{n-1}}{(n-1)!} I_{n-1}(z)$, $\therefore$ $e^z = \sum_{r=0}^{n} \frac{z^r}{r!} + \frac{z^n}{(n-1)!} I_{n-1}(z)$.

10. $I_n = \int_0^1 (1-x^2)^n dx = \int_0^1 (1-x^2)(1-x^2)^{n-1} dx = \int_0^1 (1-x^2)^{n-1} dx - \int_0^1 x^2 (1-x^2)^{n-1} dx$

$= I_{n-1} + \frac{1}{2n}\int_0^1 x d(1-x^2)^n = I_{n-1} + \frac{1}{2n}\left\{\left[x(1-x^2)^n\right]_0^1 - \int_0^1 (1-x^2)^n dx\right\} = I_{n-1} - \frac{1}{2n} I_n$ $\therefore$ $I_n = \frac{2n}{2n+1} I_{n-1}$

$I_8 = \frac{16}{17} I_7 = \frac{16 \times 14}{17 \times 15} I_6 = \ldots = \frac{16 \times 14 \times \ldots \times 2}{17 \times 15 \times \ldots \times 3} I_0 = \frac{2^{16}(8!)^2}{17!} = \underline{\underline{\frac{32768}{109395}}}$, since $I_0 = \int_0^1 dx = 1$

11. **(a)** $I_{m,n} = \int_0^1 x^m (\ln x)^n dx = \int_0^1 \frac{x^{m+1}}{n+1} d\left[(\ln x)^{n+1}\right] = \left[\frac{x^{m+1}}{n+1}(\ln x)^{n+1}\right]_0^1 - \int_0^1 \frac{(\ln x)^{n+1}}{n+1}(m+1)x^m dx$

$= -\frac{m+1}{n+1} I_{m,n+1}$ $\therefore$ $I_{m,n+1} = -\frac{n+1}{m+1} I_{m,n}$

$\therefore$ $I_{m,n} = -\frac{n}{m+1} I_{m,n} = \left(-\frac{n}{m+1}\right)\left(-\frac{n-1}{m+1}\right)\ldots\left(-\frac{1}{m+1}\right) I_{m,0} = \frac{(-1)^n n!}{(m+1)^n}\int_0^1 x^m dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$

(b) First part is similar to question 4.

$I_4 = \frac{3}{6} a^2 I_2 = \frac{1}{2} a^2 \left(\frac{1}{4} a^2 I_0\right) = \frac{1}{8} a^4 \int_0^a (a^2 - x^2)^{1/2} dx \underset{x = a\sin\theta}{=} \frac{1}{8} a^4 \int_0^{\pi/2} (a^2 - a^2 \sin^2\theta) d(a\sin\theta)$

$= \frac{1}{8} a^6 \int_0^{\pi/2} \cos^2\theta d\theta = \underline{\underline{\frac{\pi a^6}{32}}}$

12. **(a)** $I_k = \int_0^{\pi/2} \sin^k x \, dx = -\int_0^{\pi/2} \sin^{k-1} x \, d(\cos x) = -\left[\sin^{k-1} x \cos x\right]_0^{\pi/2} + \int_0^{\pi/2} (k-1)\sin^{k-2} x \cos^2 x dx$

$= (k-1)\int_0^{\pi/2} \sin^{k-2} x (1 - \sin^2 x) dx = (k-1) I_{k-2} - (k-1) I_k$ $\therefore$ $I_k = \frac{k-1}{k} I_{k-2}$ ….(1)

(i) $\int_0^{\pi/2} \sin^{2n} x \, dx = I_{2n} = \frac{2n-1}{2n} I_{2n-2} = \frac{2n-1}{2n}\frac{2n-3}{2n-2} I_{2n-4} = \ldots = \frac{2n-1}{2n}\frac{2n-3}{2n-2}\ldots\frac{1}{2} I_0$

$= \frac{(2n)!}{2^{2n} n! n!}\int_0^{\pi/2} dx = \frac{(2n)!}{2^{2n} n! n!} \frac{\pi}{2}$ ….(2)

(ii) $\int_0^{\pi/2} \sin^{2n+1} x \, dx = I_{2n} = \frac{2n}{2n+1} I_{2n-1} = \frac{2n}{2n+1}\frac{2n-2}{2n-3} I_{2n-3} = \ldots = \frac{2n}{2n+1}\frac{2n-2}{2n-3}\ldots\frac{2}{3} I_1$

$= \frac{2n}{2n+1}\frac{2n-2}{2n-3}\ldots\frac{2}{3}\int_0^{\pi/2} \sin x \, dx = \frac{2^{2n} n! n!}{(2n+1)!}\left[-\cos x\right]_0^{\pi/2} = \frac{2^{2n} n! n!}{(2n+1)!}$ ….(3)

(iii) From (1), $I_{2n-1} = \frac{2n+1}{2n} I_{2n+1} = \left(1 + \frac{1}{2n}\right)\int_0^{\pi/2} \sin^{2n+1} x dx$

(b) $\sin^{2n+1} x \le \sin^{2n} \le \sin^{2n-1} x \qquad \forall x \in \left(0.\frac{\pi}{2}\right) \qquad \therefore \qquad \int_0^{\pi/2} \sin^{2n+1} xdx \le \int_0^{\pi/2} \sin^{2n} dx \le \int_0^{\pi/2} \sin^{2n-1} xdx$

$$\therefore \qquad 1 \le \frac{\int_0^{\pi/2} \sin^{2n} dx}{\int_0^{\pi/2} \sin^{2n+1} xdx} \le 1 + \frac{1}{2n} \qquad , \qquad \text{by (a)(iii)}$$

$$\therefore \qquad \int_0^{\pi/2} \sin^{2n} x \ dx = \left(1 + \frac{\theta_n}{2n}\right) \int_0^{\pi/2} \sin^{2n+1} xdx \quad , \quad \text{where} \quad 0 < \theta_n < 1 .$$

(c) (2) × (3), $\qquad I_{2n} I_{2n+1} = \dfrac{\pi}{2(2n+1)} \qquad$ ….(4)

From (b), $\qquad I_{2n} / I_{2n+1} = 1 + \dfrac{\theta_n}{2n} = \dfrac{2n+1}{2n}\left(1 - \dfrac{1-\theta_n}{2n+1}\right) \qquad$ ….(5)

$\sqrt{(4)\times(5)}$, $\qquad I_{2n} = \int_0^{\pi/2} \sin^{2n} x \ dx = \sqrt{1 - \dfrac{1-\theta_n}{2n+1}} \sqrt{\dfrac{\pi}{n}} \cdot \dfrac{1}{2} \qquad$ ….(6)

From (2) and (6), $\qquad \dfrac{(2n)!}{2^{2n} n! n!} = \sqrt{1 - \dfrac{1-\theta_n}{2n+1}} \dfrac{1}{\sqrt{n\pi}}$

13. **(a)** $I_n = \int_0^{\pi/4} \left(\dfrac{\sin x - \cos x}{\sin x + \cos x}\right)^{2n+1} dx = -\int_0^{\pi/4} \dfrac{(\sin x - \cos x)^{2n}}{(\sin x + \cos x)^{2n+1}} d(\sin x + \cos x)$

$$= \frac{1}{2n} \int_0^{\pi/4} (\sin x - \cos x)^{2n} d(\sin x + \cos x)^{-2n}$$

$$= \frac{1}{2n} \left[(\sin x - \cos x)^{2n} (\sin x + \cos x)^{-2n} \right]_0^{\pi/4} - \int_0^{\pi/4} (\sin x + \cos x)^{-2n+1} (\sin x - \cos x)^{2n-1} dx$$

$$= -\frac{1}{2n} - \int_0^{\pi/4} \left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)^{2(n-1)+1} dx = -\frac{1}{2n} - I_{n-1}$$

$$\therefore \qquad I_{2n} = -\frac{1}{2n} - I_{n-1} = -\frac{1}{2n} + \frac{1}{2(n-1)} + I_{n-2} = \ldots = -\frac{1}{2n} + \frac{1}{2(n-1)} - \frac{1}{2(n-2)} + \ldots + (-1)^n I_0$$

$$= -\frac{1}{2n} + \frac{1}{2(n-1)} - \frac{1}{2(n-2)} + \ldots + (-1)^n \ln\sqrt{2}$$

since $I_0 = \int_0^{\pi/4} \left(\dfrac{\sin x - \cos x}{\sin x + \cos x}\right) dx = \int_0^{\pi/4} \dfrac{d(\sin x + \cos x)}{\sin x + \cos x} = \ln(\sin x + \cos x)\Big|_0^{\pi/4} = -\ln\sqrt{2}$

(b) $\int (\ln\cos x)\cos x dx = \int (\ln\cos x) d(\sin x) = (\ln\cos x)(\sin x) - \int \dfrac{-\sin x}{\cos x} \sin x dx$

$$= (\ln\cos x)(\sin x) + \int \frac{1-\cos^2 x}{\cos x} dx = (\ln\cos x)(\sin x) + \int (\sec x - \cos x) dx$$

$$= (\ln\cos x)(\sin x) + \ln(\sec x + \tan x) - \sin x + C = (\ln\cos x)(\sin x) + \ln\frac{1+\sin x}{\cos x} - \sin x + C$$

$$= (\ln\cos x)(\sin x) + \ln(1+\sin x) - \ln(\cos x) - \sin x + C = (\ln\cos x)(\sin x - 1) + \ln(1+\sin x) - \sin x + C$$

$$\lim_{x\to\frac{\pi}{2}}(\ln\cos x)(\sin x-1)=\lim_{x\to\frac{\pi}{2}}\frac{\ln\cos x}{\frac{1}{\sin x-1}}\underset{\text{LHR}}{=}\lim_{x\to\frac{\pi}{2}}\frac{-\frac{\sin x}{\cos x}}{-\frac{1}{(\sin x-1)^2}\cos x}=\lim_{x\to\frac{\pi}{2}}\sin x\left(\lim_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\cos x}\right)^2=1\times\left(\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{-\sin x}\right)^2=0$$

$$\therefore\quad \int_0^{\pi/2}(\ln\cos x)\cos x dx=0+\ln\left(1+\sin\frac{\pi}{2}\right)-[(\sin 0-1)(\ln\cos 0)+\ln(1+\sin 0)-\sin 0]=\ln 2-1$$

14. **(i)** $I(n,n-1)=\int_b^a (x-a)^n(b-x)^{n-1}dx=\int_b^a (x-a)^n d\left[-\frac{(b-x)^n}{n}\right]=\left[-(x-a)^n\frac{(b-x)^n}{n}\right]_a^b-\int_b^a n(x-a)^{n-1}dx$

$= I(n-1, n)$

(ii) $I(n,n)=\int_b^a (x-a)^n(b-x)^n dx=\int_b^a (b-x)^n d\left[\frac{(x-a)^n}{n+1}\right]=\left[(b-x)^n\frac{(x-a)^n}{n+1}\right]_a^b-\int_b^a-\frac{(x-a)^n}{n+1}n(b-x)^n dx$

$$=\frac{n}{n+1}\int_b^a (x-a)^{n+1}(b-x)^{n-1}dx$$

$$\therefore\quad (n+1)I(n,n)=n\int_b^a (x-a)^n(b-x)^{n-1}[(b-a)-(b-x)]dx$$

$$=n\left[(b-a)\int_b^a (x-a)^n(b-x)^{n-1}dx-\int_b^a (x-a)^n(b-x)^n dx\right]=n[(b-a)I(n,n-1)-I(n,n)]$$

$\therefore\quad 2(2n+1)\, I(n,n) = 2n\,(b-a)\, I(n, n-1) = n\,(b-a)\,[I(n, n-1) + I(n-1, n)]$, by (1)

$$=n(b-a)\int_b^a \left[(x-a)^n(b-x)^{n-1}+(x-a)^{n-1}(b-x)^n\right]dx=n(b-a)\int_b^a(x-a)^{n-1}(b-x)^{n-1}[(x-a)+(b-x)]dx$$

$= n(b-a)^2\, I(n-1, n-1)$

$$I(n,n)=\frac{n}{2(2n+1)}(b-a)^2 I(n-1,n-1)=\frac{n}{2(2n+1)}(b-a)^2\left[\frac{n-1}{2(2n-1)}(b-a)^2 I(n-2,n-2)\right]$$

$$=\frac{[n(n-1)]^2}{(2n+1)(2n)(2n-1)(2n-2)}(b-a)^4 I(n-2,n-2)=\ldots=\frac{n!\,n!}{(2n+1)!}(b-a)^{2n}I(0,0)=\frac{n!\,n!}{(2n+1)!}(b-a)^{2n+1}$$

since $I(0, 0) = b - a$.

15. $I(n)=\int\left(a^2+x^2\right)^{n/2}dx=x\left(a^2+x^2\right)^{n/2}-\int x\frac{n}{2}\left(a^2+x^2\right)^{n/2-1}(2x)dx=x\left(a^2+x^2\right)^{n/2}-n\int x^2\left(a^2+x^2\right)^{(n-2)/2}dx$

$$=x\left(a^2+x^2\right)^{n/2}-n\int\left[\left(a^2+x^2\right)-a^2\right]\left(a^2+x^2\right)^{(n-2)/2}dx=x\left(a^2+x^2\right)^{n/2}-nI(n)+na^2I(n-2)$$

$$\therefore\quad I(n)=\frac{1}{n+1}x\left(a^2+x^2\right)^{n/2}+\frac{n}{n+1}a^2I(n-2)$$

$$\int_0^2\left(5+x^2\right)^{3/2}dx=\frac{1}{4}x\left(5+x^2\right)^{3/2}\Big|_0^2+\frac{3}{4}5\int_0^2\left(5+x^2\right)^{1/2}dx=\frac{27}{2}+\frac{15}{4}\left[3+\frac{5}{4}\ln 5\right]=\frac{396+75\ln 5}{16}$$

since $\int_0^2\left(5+x^2\right)^{1/2}dx=\frac{x}{2}\sqrt{5+x^2}+\frac{5}{2}\ln\left(x+\sqrt{5+x^2}\right)+C$